SPRING 2025: MATH 590 HOMEWORK

The page and section numbers in the assignments below refer to those in the course textbook.

Tuesday, January 21. Section 1.1: 1, 3, 5, 6, 7 and Section 1.2: 13, 15.

Optional Bonus Problem 1. For a possible 2 bonus points, use facts from a first course in linear algebra to prove that if $W \subsetneq \mathbb{R}^2$ is a non-zero proper subspace, then W is a line through the origin. Turn this in on Thursday, January 23.

Thursday, January 23. Section 1.3: 1a, 3, 8, 9b, 11, 12.

Tuesday, January 28. Section 1.4: 1(a) - 1(f), 4, 5, 7, 9a, 10a, 12 and the following problem. Determine if the vectors $v_1 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix}$ are linearly dependent or linearly independent. If they are linearly dependent, provide a non-trivial dependence relation among them.

Thursday, January 30. Section 1.4: 2 and Chapter One Supplementary Exercises: 5, 8, 9.

Tuesday, February 4. Section 1.6: 2a,b,c,d; 15a,b; 16a.

Thursday, February 6. Section 1.6: 7a, 7c, 9a, 9b. And the following problem: Suppose V has dimension four, $V = W_1 \oplus W_2$, $S_1 = \{u, u'\}$ is a basis for W_1 and $S_2 = \{w, w'\}$ is a bases for W_2. Prove that $S_1 \cup S_2$ is a basis for V.

Tuesday, February 11. Section 3.2: 1a, 1f, 2d, And: Suppose $A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$. Use either a row or column expansion to prove $|A| = adf$.

Thursday, February 13. Section 3.3: 7a, 7b, 10 (taking $n = 3$). And the following problem:

Let $A = \begin{pmatrix} a & b & c & d \\ x & y & z & w \\ 0 & 0 & e & f \\ 0 & 0 & g & h \end{pmatrix}$, and set $C = \begin{pmatrix} a & b \\ x & y \end{pmatrix}$, $D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Show that $|A| = |C| \cdot |D|$.

Tuesday, February 18. Snow day.

Thursday, February 20. 1. Give a definition of a 3×3 elementary matrix and then write $A = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & 6 \end{pmatrix}$ as a product of elementary matrices. And, Section 4.1: 1a, 1b, 3a, 3b, 3c.

Tuesday, March 4. Section 4.2: 1b, c,d, e, 3, 6a.

Bonus Problem 4. (3 points) Two $n \times n$ matrices A, B are said to be *similar* if there exists an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. In this case we write $A \sim B$. For 3 bonus points, prove the following statements for all $n \times n$ matrices A, B, C:

(i) $A \sim A$
(ii) If $A \sim B$, then $B \sim A$.
(iii) If $A \sim B$ and $B \sim C$, then $A \sim C$.

Thursday, March 6. Chapter 4, Supplementary exercises: 4a, b, c, d and for the matrix in $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, verify that eigenvectors corresponding to the two distinct eignvalues are orthogonal. This is an important property of symmetric matrices.

Bonus Problem 5. Let A be a 7×7 matrix with entries in $\mathbb{R}$ with $p_A(x) = (x-\lambda_1)^2(x-\lambda_2)^3(x-\lambda_3)^2$. Suppose $\dim(E_{\lambda_1}) = 2, \dim(E_{\lambda_2}) = 3, \dim(E_{\lambda_3}) = 2$. Give a complete proof that A is diagonalizable, using the techniques from class. (4 points)

Tuesday, March 11. Section 4.3: 7, 10a, 13 and the following problem. Using the fact that $u_1 := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, u_2 := \frac{1}{\sqrt{3}} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 := \frac{1}{\sqrt{6}} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is an ONB for $\mathbb{R}^3$, use the inner product to write $v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ as a linear combination of u_1, u_2, u_3. Be sure to check your answer.

Thursday, March 13. 1. Find an orthogonal basis for the subspace of $\mathbb{R}^4$ generated by the three vectors $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

2. For $f(x), g(x) \in P_2(\mathbb{R})$, define $\langle f(x), g(x) \rangle := \int_0^1 f(x)g(x)\, dx$. Starting with the basis $\{1, x, x^2\}$ apply the Gram-Schmidt process to construct an orthonormal basis. Then use the inner product to write $1 + 2x + 3x^2$ in terms of the orthonormal basis you found.

Tuesday, March 25. Set $A := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$, so that A is symmetric. First diagonalize this matrix, as we have done in the past. Then note that the eigenvectors you use to diagonalize are mutually orthogonal. Change these into unit vectors. Then create an orthogonal matrix Q and use this to show that A can be orthogonally diagonalized.

Bonus Problem 6. Let W subspace of the inner product space V. Suppose $\{u_1, \ldots, u_r\}$ is an orthonormal basis for W. Extend this basis to $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_n\}$, an orthonormal basis for V. Show that $\{u_{r+1}, \ldots, u_n\}$ is a basis for $W^\perp$. (2 points)

Thursday, March 27. 1. Find an orthogonal matrix Q that orthogonally diagonalizes the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, and check that the matrix you found works. Note, for this, you can show $AQ = QD$, where D is the resulting diagonal matrix.

2. Do the same for the matrix $\begin{pmatrix} 9 & 0 & 0 & 0 \\ 0 & 8 & -2 & 2 \\ 0 & -2 & 5 & 4 \\ 0 & 2 & 4 & 5 \end{pmatrix}$.

Bonus Problem 7. Let A be the $n \times n$ matrix whose entries are all 1. Prove that A is diagonalizable. Due Tuesday, April 1. (5 points)

Tuesday, April 1. Following the example from today's lectures, find the singular value decomposition of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Thursday, April 3. Set $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}$. Find the singular value decomposition of A in two ways: First starting with A^tA, and then starting with AA^t.

2. For $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, consider the system of equations $A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$. Show that this system of equation has no solution, and then find the best approximate solution by first calculating the pseudo-inverse A^+.

Bonus Problem 8. Let C be an $r \times s$ matrix and $u \in \mathbb{R}^s$. Show that if u is in the null space of C and u^t is in the row space of C, then $u = \vec{0}$. Due Tuesday April 8. (2 points)

Tuesday, April 8. 1. Given $A = \begin{pmatrix} 1+i & -2i \\ 3 & 4_i \end{pmatrix}$, $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $w = \begin{pmatrix} 2 \\ i \end{pmatrix}$, verify properties (i)-(iv) listed under 'Properties of the adjoint' in today's lecture.

2. Apply the Gram-Schmidt process to the vectors v, w in the problem above to obtain an orthogonal basis for $\mathbf{C}^2$, then use this to get an orthonormal basis for $\mathbf{C}^2$.

Bonus Problem 9. Give a proof of properties (i)-(iv) listed under 'Properties of the adjoint' in today's lecture for 2×2 complex matrices and vectors in $\mathbf{C}^2$. Due Thursday, April 10. (4 points)

Thursday, April 10. 1. Verify that the matrix $A = \begin{pmatrix} 4i & -1+i \\ 1-i & 4i \end{pmatrix}$ is a normal matrix, but not self-adjoint, then find an 2×2 unitary matrix that diagonalizes A.

2. Show that the matrix $B = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 0 \\ -4 & 0 & 2 \end{pmatrix}$ is normal. Find an orthogonal (real) matrix Q such that $Q^{-1}BQ$ has the form $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}$. Hint: First find the only real eigenvalue of B, and a unit vector for its eigenspace. Then extend this vector to an orthonormal basis for $\mathbb{R}^3$ and let Q be the matrix whose columns are the basis you found.

Bonus Problem 10. In class we learned that symmetric matrices over $\mathbb{R}$ are precisely the matrices that are orthogonally diagonalizable, while over $\mathbf{C}$, normal matrices are precisely the matrices that are unitarily diagonalizable. What about normal matrices over $\mathbb{R}$? It turns out that given an normal matrix A over $\mathbb{R}$, there exists an orthogonal matrix Q over $\mathbb{R}$ such that $Q^{-1}AQ = C$, where C has block diagonal form, where each block is either 1×1 or a 2×2 matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, with $a, b \in \mathbb{R}$. **For two bonus points** show that if A is a 2×2 normal matrix that is not symmetric, then A already has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for $a, b \in \mathbb{R}$. Hint: Start with an arbitrary 2×2 matrix over $\mathbb{R}$ and write down what it means for it to be a normal matrix. (Due Tuesday, April 15.)

Tuesday, April 22. Find the JCF and the change of basis matrices for $A = \begin{pmatrix} 0 & 25 \\ -1 & 10 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix}$.

Thursday,, April 24. Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the two eigenvalues λ_1, λ_2. Arrange the eigenvalues so that λ_1 is the eigenvalue with algebraic multiplicity 2.
(ii) Calculate E_{λ_1}.
(ii) Find a vector v_2 in the null space of $(A - \lambda_1 I)^2$ that is not in E_{λ_1}.
(iv) Set $v_1 := (A - \lambda_1 I)v_2$.
(v) Take v_3 any eigenvector associated to λ_2.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3 verify that $P^{-1}AP = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 2 & -1 \\ 2 & 0 & 0 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.

(i) Find $p_A(x)$ and the single eigenvalue λ.

(ii) Calculate E_λ.
(iii) Find $v_2 \notin E_\lambda$.
(iv) Set $v_1 := (A - \lambda I)v_2$. This turns out to be a vector in E_λ.
(v) Take $v_3 \in E_\lambda$ not a multiple of v_1.
(vi) Letting P be the matrix whose columns are v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$.

3. Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Follow the steps below to find the JCF of A and the change of basis matrix P.
(i) Find $p_A(x)$ and the single eigenvalue λ.
(ii) Calculate E_λ.
(iii) Calculate $(A - \lambda I)^2$.
(iv) Find v_3 not in the null space of $(A - \lambda I)^2$.
(v) Take $v_2 := (A - \lambda I)v_3$ and $v_1 := (A - \lambda I)v_2$.
(vi) Letting P be the matrix whose v_1, v_2, v_3, verify that $P^{-1}AP = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$.

Tuesday, April 29. Consider the linear transformations $T : \mathbb{R}^2 \to \mathbb{R}^3$ with $T(x, y) = (2x+3y, -x+y, 4x+3y)$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ with $S(x, y, z) = (x - y + z, -x + y - z)$.

1. Show that T and S are linear transformations.

2. Letting α denote the standard basis for $\mathbb{R}^2$ and β denote the standard basis for $\mathbb{R}^3$, verify the formula $[ST]_\alpha^\alpha = [S]_\beta^\alpha \cdot [T]_\alpha^\beta$.

Bonus Problem 11. Suppose A is a non-diagonalizable 3×3 matrix with $p_A(x) = (x - \lambda_1)^2(x - \lambda_2)$, so that $\dim(E_{\lambda_1}) = 1$. Thus, the JCF of A has the form $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$. Let v_2 be any vector satisfying $(A - \lambda_1 I)^2 v_2 = 0$, but v_2 is not an eigenvector for λ_1 and set $v_1 := (A - \lambda_1 I)v_2$. Take v_3 any eigenvector of λ_2. Show that v_1, v_2, v_3 are linearly independent. Hint: Start with a dependence relation on v_1, v_2, v_3 and apply $(A - \lambda_1 I_3)^2$ to this equation. (5 points)

Thursday, May 1. 1. Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (14x - 13y + 8z, -13x + 14y + 8z, 8x + 8y - 7z).$$

Show that T is a symmetric linear transformation and find an orthonormal basis consisting of eigenvectors of T.

2. For T as in the previous problem, find a basis β for $\mathbb{R}^3$ such that $[T]_\beta^\beta$ is not symmetric.

3. Define $T : \mathbf{C}^3 \to \mathbf{C}^3$ by $T(x, y, z) = (x + y, x + y + z, -y + z)$. Find a basis $\alpha \subseteq \mathbf{C}^3$ such that $[T]_\alpha^\alpha$ is in Jordan canonical form.

Bonus Problem 12. Let V be a two dimension vector space over $\mathbb{R}$ and suppose $\alpha = \{u_1, u_2\}$ and $\beta = \{v_1, v_2\}$ are orthonormal basis. Let $T : V \to V$ be a linear transformation, and set $A = [T]_\alpha^\alpha$ and $B = [T]_\beta^\beta$, so that by the change of basis formula, $B = P^{-1}AP$, for an invertible 2×2 matrix P. Prove that P is an orthogonal matrix. Conclude that if the matrix of T with respect to some orthonormal basis is symmetric, then the matrix of T with respect to every orthonormal basis is symmetric. (5 points)